Analysis of Thermal Constriction Resistance with Adiabatic Circular Gaps

Kek-Kiong Tio* and Satwindar S. Sadhal†
University of Southern California, Los Angeles, California 90089

In an attempt to generalize the thermal constriction resistance functions for various contact geometries, we have obtained results pertaining to arrays of adiabatic circular regions on an otherwise isothermal surface. In particular, the steady-state heat conduction problem of a semi-infinite solid with square and hexagonal arrays of adiabatic disks on the surface is treated in detail. An analytical corrective-iterative method is applied to solve these problems of spatially periodic mixed Dirichlet and Neumann conditions at the solid surface. The thermal constriction resistance is given by expressions involving a power series of $\kappa'^{1/2}$, where κ' is the fraction of the interfacial area occupied by the adiabatic disks. By using the corresponding results for isothermal disks and combining them with the current results of adiabatic disks, a simple approximate generalized formula is developed for the various types of periodic arrangements. The following expression for the dimensional resistance is found to be quite good as far as discrete circular contacts or gaps are concerned:

$$\Phi^* = \left[\pi(1-\kappa)^2\left\{1-\left[1-16/(3\pi^2)\right]\kappa^{1/2}\right\}\right]/(2kp)$$

where p is the perimeter of the contact or the gap region per unit area of interface, k the thermal conductivity, and $\kappa = (1-\kappa')$ the area fraction in contact.

Nomenclature

A	= area of a unit cell
A_c	= area of the contact region of a unit cell
b	= distance between the reference disk and any other disk, Fig. 1
$h_i, h_{9,6}, H_i, H_{9,6}$	= array sums, Eqs. (44-46)
k	= thermal conductivity
L, l	= length scales to nondimensionalize resistance, Eqs. (3), (4), and (51)
N	= a positive integer
P	= perimeter of the contact/gap region of a unit cell
$P_n(\cdot)$	= Legendre polynomial of degree n
p	= perimeter of the contact/gap region per unit area of interface
Q	= total rate of heat flow across a unit cell
$Q_n(\cdot)$	= Legendre function of the second kind
$q, \bar{q}$	= heat flux, average surface heat flux
$\mathbf{q}, \bar{\mathbf{q}}$	= heat flux, average surface heat flux vectors
R	= disk radius
(r,ϕ,z)	= cylindrical coordinates, origin at the center of the reference disk
(r,θ)	= polar coordinates with θ measured from the line joining the reference disk and the neighboring disk under consideration, Fig. 1
$s_i, s_{i,4}, S_i, S_{i,4}$	= array sums, Eqs. (28), (29), and (35)
$T, \tilde{T}$	= temperature
$\hat{z}$	= unit vector in the z direction
(α,β)	= oblate spheroidal coordinates, Eq. (12)
ΔT	= difference between contact and average surface temperatures
ΔT^*	= difference between contact and gap temperatures
$\overline{\Delta T^*}$	= ΔT^* averaged over a disk
δ	= distance between a disk and its closest neighbor
κ, κ'	= area fraction of contact, disk area fraction
Ξ_p	= Eq. (34)
(ρ,ϕ,z)	= cylindrical coordinates with the origin at the center of an arbitrary disk and ϕ measured from the x axis, Figs. 2 and 3
Φ^*, ψ^*	= constriction resistance, Eqs. (1) and (2)
Φ, ψ	= dimenisonless resistance, Eqs. (7) and (8)
Φ_P, ψ_P	= dimensionless resistance, Eqs. (5) and (6)
Ψ	= dimensionless resistance, Eq. (50)

Subscripts

A	= reference disk
H	= hexagonal array
S	= square array
∞	= region deep inside the solid

Superscripts

(i)	= of order $(R/\delta)^i$
$(i+j)$	= $T^{(i+j)}$ superposed at the reference disk will remove the heat flux of order $(i+j)$ contributed by the jth order expansions of the $T^{(i)}$ fields associated with the surrounding disks
$(i),j$	= of order $(R/\delta)^i$ and angular dependence of $\cos j\phi$
$(k+l+m)$	= similar to $(i+j)$, $k+l$ and m replacing i and j, respectively

Received Nov. 3, 1989; revision received April 9, 1990; accepted for publication April 10, 1990.

*Graduate Student, Department of Mechanical Engineering.

†Associate Professor, Department of Mechanical Engineering. Member AIAA.

I. Introduction

THE study of thermal contact conductance dates back at least four decades. Since extensive reviews are available,[1-3] the survey of the literature presented below is quite selective and is intended as an introduction leading to the present study only.

One of the simplest problems in contact heat transfer is that of the steady-state heat conduction problem of two semi-infinite solids in contact over a single circular area. This problem readily reduces to the classical problem of the isothermal disk on a semi-infinite solid or the "electrified disk" problem. It appears that the first solution was given by Weber[4] and is available in textbooks (see e.g., Jackson[5]). The problem of transient heat conduction was attacked by Heasley,[6] who obtained an approximate solution by assuming the region of contact to be a perfectly conducting sphere. Later, Schneider et al.[7] solved the transient problem numerically, and Sadhal[8] obtained an analytical solution by using a long-time perturbation scheme. Using the "unsteady surface element" method, Beck and Keltner[9] obtained analytical solutions valid for short-time and long-time regimes. The problem of a semi-infinite solid heated over a circular disk on its surface has also been examined by a number of investigators. Thomas[10] and Beck[11] studied the transient problem of a disk heated by a uniform flux. Norminton and Blackwell[12] obtained a large-time transient solution to the problem of an isothermal thin disk immersed in an infinite medium; later, Blackwell[13] calculated the small-time solution. The steady-state problem of a disk heated by different fluxes was treated numerically by Strong et al.[14] A numerical solution was also employed by Marder and Keltner[15] in treating a disk with a step change in temperature. Recently, the problem of a circular disk on a semi-infinite solid with convective boundary conditions applied over the disk or the surrounding region was taken up by Gladwell et al.[16] and by Lemczyk and Yovanovich.[17] The problem of a semi-infinite solid heated by an arbitrary flux distribution over an arbitrarily shaped contact area on the surface has also been examined.[18] The corresponding Dirichlet problem was treated by Schneider.[19]

The problem of multiple contacts has also been examined by a number of investigators. In this regard, Dundurs and Panek[20] obtained an exact solution to the two-dimensional steady-state problem of two semi-infinite solids in contact over a series of equally spaced strips. The corresponding transient problem was treated by Sadhal,[21] who obtained the large-time asymptotic solution. Greenwood[22] considered the problem of a single cluster of contacts on a half-space and calculated the constriction resistance by taking into account the interactions of the contact spots. In a more recent study, Beck[23] calculated the thermal constriction resistance of multiple circular contacts arranged in a square array on a half-space by taking into account the interactions among the disks, which were heated by a uniform flux.

For circular contacts arranged in a square array on a half-space, the problem can be reduced to that of one contact on a semi-infinite rectangular prism. Sadhal[24] obtained the exact solutions for steady and unsteady conduction in a rectangular prism with an elliptical contact heated by different fluxes. Using the method of "optimized images," Negus and Yovanovich[25] calculated the thermal constriction resistance for a square prism heated by different fluxes over a circular contact on top of the prism. In a subsequent development, Negus et al.[26] treated a rectangular array of elliptical contacts on a semi-infinite solid. Whereas the major (or minor) axes of all the ellipses were parallel, they had an arbitrary orientation to the line joining the centers of the ellipses. By using the "equivalent flux" for a single isothermal ellipse, the authors obtained expressions for the constriction resistance. One of the principal contributions here was the delineation of the effect of the orientation of the ellipses.

The approximation of the multiple-contact situation with a cylindrical cell has been quite popular owing to the axial symmetry. There have been numerous studies of the equipotential disk in an infinite insulated cylinder.[27-29] In a very recent development with the cylindrical cell, Gladwell and Lemczyk[30] considered a broad class of mixed boundary conditions for a finite cylinder.

In this paper, we study the thermal constriction resistance at the interface of two semi-infinite solids in partial contact. The contact model consists of steady-state thermal contact over the interface, except at discrete circular gaps arranged periodically throughout the interface. This is in contrast from the usual model of discrete contact areas. The purpose of the present analysis is mainly to see the effect of the change in contact geometry in order to develop some kind of generalized formulas to represent the resistance.

Situations involving spatially periodic contact areas arise in special cases such as when we have machined surfaces. In general, however, the contacts have random shape, size, and distribution. This is a considerably more difficult problem, and an exact solution does not appear to be immediately feasible. A first-order correction for randomly placed circular contacts can, however, be worked out by Batchelor's[31] method with a great deal of lengthy algebra. An easier, but approximate, way to handle the random problem would perhaps be some kind of a self-consistent scheme. This is presently under examination. The problem with such schemes usually is that it is difficult to estimate the error. The other alternative is a completely numerical solution.

By utilizing the symmetry of the geometry and the uniqueness of the solution, apart from an additive constant, it can be shown that the region of the interface in contact is isothermal. Consequently, the original problem of two solids coupled by the conditions of continuity of temperature and heat flux at the region of thermal contact is reducible to that of one solid. In the remainder of this paper, we will treat one solid only. Thus, we consider the steady-state conduction in a semi-infinite solid. At the solid surface, the no-flux boundary condition is prescribed over the periodically arranged disks, while the region outside the disks is kept isothermal. Two cases of disk arrangement will be treated: square array and hexagonal array.

A commonly used definition of constriction resistance is

$$\Phi^* \equiv \Delta T/\bar{q} \tag{1}$$

Here, the conductance $h = 1/\Phi^*$ would correspond to a heat transfer coefficient. We also have another definition of the constriction resistance based on the total rate of heat flow Q across a unit cell into the solid:

$$\psi^* \equiv \Delta T/Q = \Phi^*/A \tag{2}$$

To nondimensionalize Φ^* and ψ^*, we need a length scale l, so that

$$\Phi \equiv (4k\Phi^*)/l \tag{3}$$

$$\psi \equiv 4kl\psi^* \tag{4}$$

For periodically arranged disks, either definition of the resistance can be used. However, for randomly placed disks, we would not have any cells to speak of and hence we cannot define Q properly. In the case of discrete contact areas, we may think of an average Q crossing every contact area. But this cannot be easily generalized to the current problem of discrete gaps. Bearing in mind that $\bar{q}$ is simple to define for nearly all types of situations, Eq. (1) is more suitable. The important question concerning the length scale l still remains. We need to find a length scale that is sufficiently general so that it is applicable to a broad class of contact geometries. Careful examination has shown that the perimeter P of the contact region or the gap can be a quite useful length scale. One of the reasons why this is considered desirable is that the leading-order calculation for isothermal elliptical contacts shows the resistance being inversely proportional to P. We may therefore define the dimensionless resistances as follows:

$$\Phi_P \equiv \frac{4k\Phi^*}{P/2\pi} \tag{5}$$

$$\psi_P \equiv 4k\psi^*(P/2\pi) = 4k\Phi^*(p/2\pi) \tag{6}$$

We can define $\bar{p}$ as the average of p and use the former to provide sufficient generality. Nevertheless, a length scale based on the perimeter alone can be somewhat misleading because not only does it vary with the area fraction in contact but it increases with κ for circular contacts while it decreases for gaps. Because it does not fit properly into the global perspective of circular gaps and contacts, we consider A_c and use its square root as a length scale. This length scale is consistently monotonic with κ. Clearly, it is useful only for periodic problems, but because the present development deals with such problems, this form is acceptable. Hence, we define

$$\Phi \equiv 4k(\pi A_c)^{-1/2}\Phi^* \tag{7}$$

$$\psi \equiv 4k[(1/\pi)A_c]^{1/2}\psi^* = \Phi A_c/A = \Phi\kappa \tag{8}$$

We point out here that the usefulness of $A_c^{1/2}$ as a scaling parameter has also been noted by Negus et al.[32] for the uniform flux condition in a variety of geometries. It had been implicitly used by Sadhal[24] in defining a length scale for elliptical contacts in rectangular arrays. The attractiveness of P as a length scale, however, still remains to be realized. In view of the above-noted fact that the exact solution for the single isothermal ellipse gives the heat flow as being proportional to P, we have pursued this line of thought with persistence. In Section III we incorporate P and κ into a suitable length scale that fits very well into an overall picture for the isothermal condition at the contacting regions.

II. Solution

The steady temperature distribution in the solid is described by Laplace's equation

$$\nabla^2 T = 0 \tag{9}$$

At the surface, $z = 0$, the boundary conditions are

$$\begin{cases} \dfrac{\partial T}{\partial z} = 0 & \text{over the disks} \\ T = \text{constant} & \text{outside the disks} \end{cases} \tag{10}$$

Deep inside the solid, i.e., $z \to \infty$,

$$-k\frac{\partial T}{\partial z} = \text{const} \tag{11}$$

The method of solution consists of solving the problem for one adiabatic disk and linearly superposing its solution to all of the other disks. As a consequence, the first disk will no longer be adiabatic. We then correct for this situation and also apply the correction to all the other disks. This is carried out iteratively until the adiabatic condition is satisfied on all the disks to a high order of accuracy. This is an extension of Beck's[23] method that Tio[33] has applied to the problem of arrays of isothermal disks.

A. Analysis of One Insulated Disk

The problem of an insulated disk on the otherwise isothermal surface of a semi-infinite solid can be solved exactly by utilizing the oblate spheroidal coordinates (α,β,ϕ), which are related to the cylindrical coordinates (ρ,ϕ,z) by

$$\rho = R\cosh\alpha\sin\beta, \qquad z = R\sinh\alpha\cos\beta$$
$$(0 \le \alpha < \infty,\ 0 \le \beta \le \pi) \tag{12}$$

In the cylindrical coordinate system, the disk is described by $z = 0$ and $0 \le \rho \le R$ and the solid by $z \ge 0$ and $0 \le \rho < \infty$. A suitable form for the steady temperature distribution in the solid is

$$T^{(0)} = -\frac{qR}{k}\frac{z}{R} + \frac{2}{\pi}\frac{qR}{k}Q_1(i\sinh\alpha)P_1(\cos\beta) \tag{13}$$

where P_1 and Q_1 are the Legendre polynomial and Legendre function of the second kind, respectively, and $i = \sqrt{-1}$. $T^{(0)}$ as given by Eq. (13) satisfies Laplace's equation, and the boundary conditions of zero flux over the disk, zero surface temperature outside the disk, and uniform flux far away from the disk. At the solid surface, the temperature is given by

$$\begin{cases} T^{(0)}|_{z=0,\,\rho<R} = -\dfrac{2}{\pi}\dfrac{qR}{k}\left[1-\left(\dfrac{\rho}{R}\right)^2\right]^{1/2} \\ T^{(0)}|_{z=0,\,\rho>R} = 0 \end{cases} \tag{14}$$

The heat flux into the solid is

$$\begin{cases} q^{(0)}|_{\text{disk}} = 0 \\ q^{(0)}|_{\rho>R} = q - \dfrac{2}{\pi}q\left[\arcsin\left(\dfrac{R}{\rho}\right) - \left(\dfrac{R}{\rho}\right)\left\{1-\left(\dfrac{R}{\rho}\right)^2\right\}^{-1/2}\right] \end{cases} \tag{15}$$

Deep inside the solid, i.e., $z \to \infty$, we have a uniform flux $\mathbf{q} = q\hat{z}$, where $\hat{z}$ is the unit vector in the z direction.

The corrective-iterative procedure employed in Ref. 33 began with the analysis of one isothermal disk. However, with the present problem, we cannot begin the procedure with a single adiabatic disk because far away from the disk, the surface heat flux does not vanish [see Eq. (15)]. Therefore, the leading term will include $\tilde{T}_S^{(0)}$ or $\tilde{T}_H^{(0)}$ for a square array or a hexagonal array, respectively, in addition to $T^{(0)}$ associated with an adiabatic disk.

The problem of $\tilde{T}_S^{(0)}$ for an isolated disk on the surface of a semi-infinite solid is governed by

$$\nabla^2\tilde{T}_S^{(0)} = 0, \qquad 0 \le \rho < \infty,\ 0 < z < \infty \tag{16}$$

and the boundary conditions

$$z = 0: \quad \left.\frac{\partial\tilde{T}_S^{(0)}}{\partial z}\right|_{\rho<R} = [(2N+1)^2 - 1]\frac{q}{k} \tag{17}$$

$$\tilde{T}_S^{(0)}|_{\rho>R} = 0 \tag{18}$$

$$(\rho^2+z^2)^{1/2} \to \infty: \quad \tilde{T}_S^{(0)} = 0 \tag{19}$$

In Eq. (17), N is a positive integer of which the significance will be evident later. By separation of variables in the oblate spheroidal coordinate system, the solution is obtained as

$$\tilde{T}_S^{(0)} = [(2N+1)^2 - 1]\frac{2}{\pi}\frac{qR}{k}Q_1(i\sinh\alpha)P_1(\cos\beta) \tag{20}$$

The surface temperature is given by

$$\begin{cases} \tilde{T}_S^{(0)}|_{z=0,\,\rho<R} = -\dfrac{2}{\pi}[(2N+1)^2-1]\dfrac{qR}{k}\left[1-\left(\dfrac{\rho}{R}\right)^2\right]^{1/2} \\ \tilde{T}_S^{(0)}|_{z=0,\,\rho>R} = 0 \end{cases} \tag{21}$$

and the surface heat flux into the solid by

$$\begin{cases} \tilde{q}_S^{(0)}|_{\text{disk}} = -[(2N+1)^2-1]q \\ \tilde{q}_S^{(0)}|_{\rho>R} = -\dfrac{2}{\pi}[(2N+1)^2-1]q \\ \quad \times\left[\arcsin\left(\dfrac{R}{\rho}\right) - \left(\dfrac{R}{\rho}\right)\left\{1-\left(\dfrac{R}{\rho}\right)^2\right\}^{-1/2}\right] \end{cases} \tag{22}$$

Deep inside the solid, i.e., $z \to \infty$, the condition of zero heat flux prevails.

For the problem of $\tilde{T}_H^{(0)}$, we replace the factor $[(2N+1)^2 - 1]$ with $[3N(N+1)]$. In the two-disk analysis below, we have in mind the case of a square array. It can be easily changed to that of a hexagonal array by replacing $\tilde{T}_S^{(0)}$ with $\tilde{T}_H^{(0)}$.

B. Analysis of Two Circular Disks

Suppose that we superpose $T^{(0)}$ and $\tilde{T}_S^{(0)}$ at each of the two disks on the surface of a semi-infinite solid (Fig. 1). Then, the region of the solid surface outside the disks is at zero temperature. At disk A, the temperature is given by

$$T_A = -\frac{2}{\pi}(2N+1)^2 \frac{qR}{k}\left[1 - \left(\frac{r}{R}\right)^2\right]^{1/2} \tag{23}$$

and the heat flux into the solid by

$$q_A = -\left[(2N+1)^2 - 1\right] q + q + \frac{2}{\pi}(2N+1)^2 q \times \left[\frac{1}{3}\left(\frac{R}{\rho}\right)^3 + \frac{3}{10}\left(\frac{R}{\rho}\right)^5 + \frac{15}{56}\left(\frac{R}{\rho}\right)^7 + \frac{35}{144}\left(\frac{R}{\rho}\right)^9 + \cdots\right] \tag{24}$$

The first term in the right side of Eq. (24) is contributed by the pair of $T^{(0)}$ and $\tilde{T}_S^{(0)}$ associated with disk A, whereas the rest of the right side is the result of the pair associated with disk B.

Using the Legendre polynomial identities,

$$\left(\frac{R}{\rho}\right)^l = \frac{1}{1\cdot 3\cdots(l-2)}\left(\frac{R}{b}\right)^l \sum_{n=(l-1)/2}^{\infty}\left(\frac{r}{b}\right)^{n-(l-1)/2} \times \left.\frac{\mathrm{d}^{(l-1)/2}P_n(x)}{\mathrm{d}x^{(l-1)/2}}\right|_{x=\cos\theta} \tag{25}$$

where $l = 3,5,7,\ldots$, we expand Eq. (24) about the center of disk A and obtain

$$\begin{aligned}
q_A = &-\left[(2N+1)^2 - 1\right] q + q + \frac{2}{\pi}(2N+1)^2 q \\
&\times \frac{1}{3}\left(\frac{R}{b}\right)^3 + \frac{2}{\pi}(2N+1)^2 q\left[\left(\frac{r}{R}\right)\cos\theta\right]\left(\frac{R}{b}\right)^4 \\
&+ \frac{2}{\pi}(2N+1)^2 q\left[\frac{3}{10} + \frac{1}{4}(5\cos 2\theta + 3)\left(\frac{r}{R}\right)^2\right]\left(\frac{R}{b}\right)^5 \\
&+ \frac{2}{\pi}(2N+1)^2 q\left[\frac{3}{2}\left(\frac{r}{R}\right)\cos\theta + \frac{5}{24}(7\cos 3\theta + 9\cos\theta) \right. \\
&\left. \times \left(\frac{r}{R}\right)^3\right]\left(\frac{R}{b}\right)^6 + \frac{2}{\pi}(2N+1)^2 q \\
&\times \left[\frac{15}{56} + \frac{3}{8}(7\cos 2\theta + 5)\left(\frac{r}{R}\right)^2 \right. \\
&\left. + \frac{5}{64}(21\cos 4\theta + 28\cos 2\theta + 15)\left(\frac{r}{R}\right)^4\right]\left(\frac{R}{b}\right)^7 \\
&+ \frac{2}{\pi}(2N+1)^2 q\left[\frac{15}{8}\left(\frac{r}{R}\right)\cos\theta + \frac{21}{16}(3\cos 3\theta + 5\cos\theta) \right. \\
&\left. \times \left(\frac{r}{R}\right)^3 + \frac{7}{128}(33\cos 5\theta + 45\cos 3\theta + 50\cos\theta)\left(\frac{r}{R}\right)^5\right] \\
&\times \left(\frac{R}{b}\right)^8 + \frac{2}{\pi}(2N+1)^2 q\left[\frac{35}{144} + \frac{15}{32}(9\cos 2\theta + 7)\left(\frac{r}{R}\right)^2 \right. \\
&+ \frac{21}{128}(33\cos 4\theta + 60\cos 2\theta + 35)\left(\frac{r}{R}\right)^4 \\
&+ \frac{7}{1536}(429\cos 6\theta + 594\cos 4\theta + 675\cos 2\theta + 350) \\
&\left. \times \left(\frac{r}{R}\right)^6\right]\left(\frac{R}{b}\right)^9 + \mathcal{O}\left(\frac{R}{b}\right)^{10}
\end{aligned} \tag{26}$$

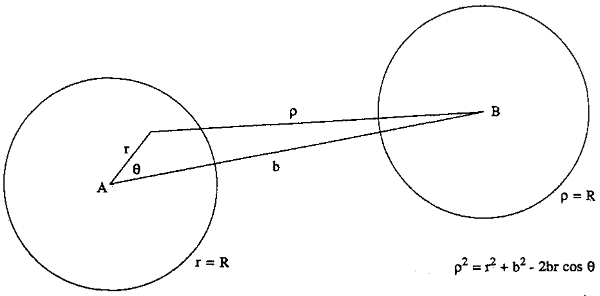

Fig. 1 Interaction of two disks of radius R.

Again, we note that the first term in the right side of Eq. (26) is contributed by disk A itself, whereas the remaining terms are due to the neighboring disk B. Thus, if we incorporate all of the surrounding disks inside the square in Fig. 2 into Eq. (26), the first two terms of the right side cancel out since there are $[(2N+1)^2 - 1]$ disks, excluding A, inside the square. The corrective-iterative procedure used in the analysis of the isothermal-disks problem[33] can then be applied to the remaining terms.

C. Adiabatic Disks in a Square Array

At each of the disks inside the square on the surface of the semi-infinite solid (Fig. 2), we superpose the temperature fields $T^{(0)}$ and $\tilde{T}_S^{(0)}$. Then, summing Eq. (26), the first term in the right side being excluded, over each of the surrounding disks, we obtain the heat flux at the reference disk as

$$\begin{aligned}
q_A = &\frac{2}{\pi}(2N+1)^2 q \cdot \frac{1}{3} s_3(N)\left(\frac{R}{\delta}\right)^3 + \frac{2}{\pi}(2N+1)^2 q \\
&\times \left[\frac{3}{10} + \frac{3}{4}\left(\frac{r}{R}\right)^2\right] s_5(N)\left(\frac{R}{\delta}\right)^5 + \frac{2}{\pi}(2N+1)^2 q \\
&\times \left\{\left[\frac{15}{56} + \frac{15}{8}\left(\frac{r}{R}\right)^2 + \frac{75}{64}\left(\frac{r}{R}\right)^4\right] s_7(N) \right. \\
&\left. + \frac{105}{64}\left(\frac{r}{R}\right)^4 s_{7,4}(N)\cos 4\phi\right\}\left(\frac{R}{\delta}\right)^7 + \frac{2}{\pi}(2N+1)^2 q \\
&\times \left\{\left[\frac{35}{144} + \frac{105}{32}\left(\frac{r}{R}\right)^2 + \frac{735}{128}\left(\frac{r}{R}\right)^4 + \frac{1225}{768}\left(\frac{r}{R}\right)^6\right] s_9(N) \right. \\
&\left. + \left[\frac{693}{128}\left(\frac{r}{R}\right)^4 + \frac{693}{256}\left(\frac{r}{R}\right)^6\right] s_{9,4}(N)\cos 4\phi\right\}\left(\frac{R}{\delta}\right)^9 \\
&+ \mathcal{O}\left(\frac{R}{\delta}\right)^{11}
\end{aligned} \tag{27}$$

where

$$s_i(N) = \left(4 + 2^{(4-i)/2}\right)\sum_{n=1}^{N}\frac{1}{n^i} + 8\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\frac{1}{(m^2+n^2)^{i/2}} \tag{28}$$

$$s_{i,4}(N) = \left(4 - 2^{(4-i)/2}\right)\sum_{n=1}^{N}\frac{1}{n^i} + 8\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\frac{1}{(m^2+n^2)^{i/2}} \times \left[2\left(\frac{m^2-n^2}{m^2+n^2}\right)^2 - 1\right] \tag{29}$$

The temperature of the reference disk is given by

$$T_A = -\frac{2}{\pi}(2N+1)^2 \frac{qR}{k}\left[1 - \left(\frac{r}{R}\right)^2\right]^{1/2} \tag{30}$$

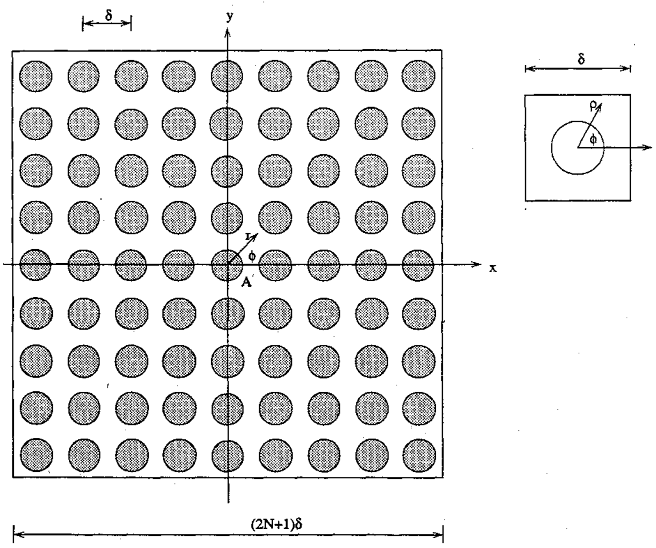

Fig. 2 Square array of adiabatic disks; the disk A at the center is the reference disk; also shown is a typical unit cell and its associated disk.

Outside the disks, the surface temperature is zero. Deep inside the solid, i.e., $z \to \infty$, there exists a uniform heat flux,

$$q_\infty = (2N+1)^2 q\hat{z} \tag{31}$$

where $\hat{z}$ is the unit vector in the z direction. Therefore, there is an average heat flux,

$$\bar{q} = (2N+1)^2 q\hat{z} \tag{32}$$

at the solid surface.

As $N \to \infty$, T_A and q_A as given by Eqs. (30) and (27), respectively, become unbounded. However, since T_A and q_A increase at the same rate as the average heat flux at the surface, they can be normalized with respect to the average flux, yielding finite values.

With only $T^{(0)}$ and $\tilde{T}_S^{(0)}$ attached to each disk, Eq. (27) shows that leakage of heat at disk A occurs as early as $\mathcal{O}(R/\delta)^3$. However, the leakage can be moved to a higher order by superposing a few more components, in addition to $T^{(0)}$ and $\tilde{T}_S^{(0)}$, at each of the disks inside the square. Thus, the term of $\mathcal{O}(R/\delta)^3$ in Eq. (27) can be removed by superposing the temperature field $T_S^{(3)}$ on $T^{(0)}$ and $\tilde{T}_S^{(0)}$ at disk A. For an isolated disk on the surface of the solid, $T_S^{(3)}$ satisfies Eqs. (16–19) with Eq. (17) replaced by the appropriate Neumann condition. The exact form of $T_S^{(3)}$ is similar to that of $\tilde{T}_S^{(0)}$, and its associated surface fluxes and disk temperature are listed in the Appendix.

Now, superposing $T_S^{(3)}$ on $T^{(0)}$ and $\tilde{T}_S^{(0)}$ at disk A removes the $\mathcal{O}(R/\delta)^3$ term from Eq. (27). However, it can be seen that the $T_S^{(3)}$ fields associated with the surrounding disks introduce fluxes of $\mathcal{O}(R/\delta)^6$ to disk A. Thus, heat leakage at disk A has been moved to $\mathcal{O}(R/\delta)^5$. To move it to order 10, we superpose the temperature fields $T_S^{(5)}$, $T_S^{(3+3)}$, $T_S^{(7)}$, $T_S^{(7),4}$, $T_S^{(3+5)}$, $T_S^{(5+3)}$, $T_S^{(9)}$, $T_S^{(9),4}$, and $T_S^{(3+3+3)}$ on $T^{(0)}$, $\tilde{T}_S^{(0)}$, and $T_S^{(3)}$ at each of the disks inside the square. For an isolated disk, these additional fields satisfy Eqs. (16–19), with Eq. (17) being replaced by the appropriate Neumann condition. As with $T_S^{(3)}$, exact forms of these temperature fields can be obtained in terms of the oblate spheroidal coordinates (α,β,ϕ). In the Appendix, we list the surface fluxes and the disk temperatures associated with these temperature fields.

Thus, superposition of $T^{(0)}$, $\tilde{T}_S^{(0)}$, $T_S^{(3)}$, $T_S^{(5)}$, $T_S^{(3+3)}$, $T_S^{(7)}$, $T_S^{(7),4}$, $T_S^{(3+5)}$, $T_S^{(5+3)}$, $T_S^{(9)}$, $T_S^{(9),4}$, and $T_S^{(3+3+3)}$ at each of the disks inside the square followed by an extensive amount of algebra yields $q_A = \mathcal{O}(R/\delta)^{10}$. Deep inside the solid, the heat flux is still governed by Eq. (31). Now, we let $N \to \infty$ and recover the case of an infinite square array of insulated disks on the surface of a semi-infinite solid. At the surface, the region external to the disks is isothermal, and the difference between this isothermal temperature and the temperature of a typical disk is given by

$$\begin{aligned}
\Delta T^* ={}& \frac{2}{\pi}\frac{\bar{q}R}{k}\Xi_\rho^{1/2} + \frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{2S_3}{3\pi}\Xi_\rho^{1/2}\left(\frac{R}{\delta}\right)^3 + \frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{S_5}{\pi} \\
&\times\left(\frac{8}{5}\Xi_\rho^{1/2} - \frac{2}{3}\Xi_\rho^{3/2}\right)\left(\frac{R}{\delta}\right)^5 + \frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{4S_3^2}{9\pi^2}\Xi_\rho^{1/2}\left(\frac{R}{\delta}\right)^6 \\
&+ \frac{2}{\pi}\frac{\bar{q}R}{k}\left\{\frac{S_7}{\pi}\left[\frac{30}{7}\Xi_\rho^{1/2} - \frac{10}{3}\Xi_\rho^{3/2} + \frac{2}{3}\Xi_\rho^{5/2}\right]\right. \\
&\left.+ \frac{4S_{7,4}}{3\pi}\Xi_\rho^{1/2}\left(\frac{\rho}{R}\right)^4\cos 4\phi\right\}\left(\frac{R}{\delta}\right)^7 + \frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{S_3S_5}{\pi^2} \\
&\times\left(\frac{28}{15}\Xi_\rho^{1/2} - \frac{4}{9}\Xi_\rho^{3/2}\right)\left(\frac{R}{\delta}\right)^8 + \frac{2}{\pi}\frac{\bar{q}R}{k}\left\{\frac{S_9}{\pi}\left[\frac{112}{9}\Xi_\rho^{1/2}\right.\right. \\
&\left. - 14\Xi_\rho^{3/2} + \frac{28}{5}\Xi_\rho^{5/2} - \frac{2}{3}\Xi_\rho^{7/2}\right] + \frac{8S_3^3}{27\pi^3}\Xi_\rho^{1/2} + \frac{S_{9,4}}{\pi} \\
&\left.\times\left(\frac{32}{5}\Xi_\rho^{1/2} - \frac{4}{3}\Xi_\rho^{3/2}\right)\left(\frac{\rho}{R}\right)^4\cos 4\phi\right\}\left(\frac{R}{\delta}\right)^9 + \mathcal{O}\left(\frac{R}{\delta}\right)^{10}
\end{aligned} \tag{33}$$

where

$$\Xi_\rho = 1 - \left(\frac{\rho}{R}\right)^2 \tag{34}$$

$$S_i = \lim_{N\to\infty} s_i(N), \qquad S_{i,4} = \lim_{N\to\infty} s_{i,4}(N) \tag{35}$$

and $\bar{q}$ is the average heat flux into the solid at its surface. The expression for ΔT^* as given by Eq. (33) corresponds to the condition of disks adiabatic up to $\mathcal{O}(R/\delta)^9$. Averaging ΔT^* over an entire disk, we obtain

$$\begin{aligned}
\overline{\Delta T^*} ={}& \frac{2}{\pi}\frac{\bar{q}R}{k}\left\{\frac{2}{3} + \frac{4S_3}{9\pi}\left(\frac{R}{\delta}\right)^3 + \frac{4S_5}{5\pi}\left(\frac{R}{\delta}\right)^5 + \frac{8S_3^2}{27\pi^2}\left(\frac{R}{\delta}\right)^6\right. \\
&+ \frac{12S_7}{7\pi}\left(\frac{R}{\delta}\right)^7 + \frac{16S_3S_5}{15\pi^2}\left(\frac{R}{\delta}\right)^8 + \left(\frac{16S_3^3}{81\pi^3} + \frac{112S_9}{27\pi}\right) \\
&\left.\times\left(\frac{R}{\delta}\right)^9 + \mathcal{O}\left(\frac{R}{\delta}\right)^{10}\right\}
\end{aligned} \tag{36}$$

The difference between the temperature of the isothermal region and the average temperature of the entire surface of the solid is given by

$$\Delta T = \frac{\pi R^2}{\delta^2}\overline{\Delta T^*} = \kappa'\overline{\Delta T^*} \tag{37}$$

where $\kappa' = \pi R^2/\delta^2$ is the area fraction of the disks. Substituting Eq. (37) into Eq. (2) and using the definition of κ', we obtain

$$\begin{aligned}
\psi^* ={}& \frac{16}{3\pi^2}\frac{1}{4Rk}\cdot\kappa'^2\left\{1 - \frac{2S_3}{3\pi^{5/2}}\kappa'^{3/2} - \frac{6S_5}{5\pi^{7/2}}\kappa'^{5/2}\right. \\
&\left. - \frac{18S_7}{7\pi^{9/2}}\kappa'^{7/2} - \frac{56S_9}{9\pi^{11/2}}\kappa'^{9/2} - \cdots\right\}^{-1}
\end{aligned} \tag{38}$$

Substituting the numerical values of $S_3 = 9.03362$, $S_5 = 5.09026$, $S_7 = 4.42312$, and $S_9 = 4.19127$ into Eq. (38) then yields

$$\begin{aligned}
\psi^* ={}& \frac{1}{4k}\left(\frac{1}{\pi}A_c\right)^{-1/2}\frac{\kappa'^{3/2}(1-\kappa')^{1/2}}{1.8506}\times\left\{1 - 0.3443\kappa'^{3/2}\right. \\
&\left. - 0.1111\kappa'^{5/2} - 0.06588\kappa'^{7/2} - 0.04808\kappa'^{9/2} - \cdots\right\}^{-1}
\end{aligned} \tag{39}$$

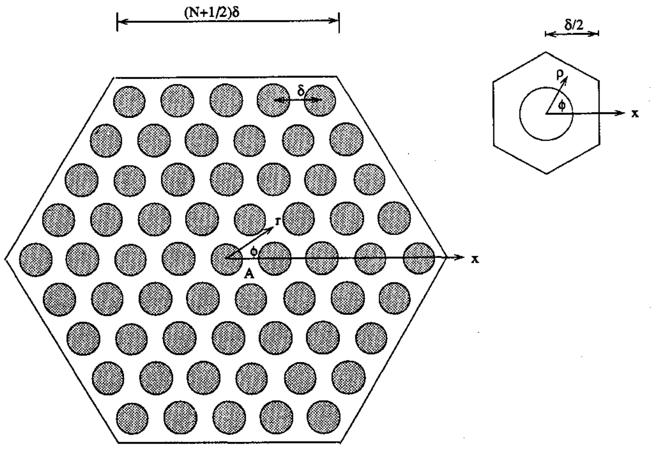

Fig. 3 Hexagonal array of adiabatic disks; the disk A at the center is the reference disk; a typical unit cell is also shown.

In Eq. (39), A_c is the area of the isothermal region (outside the disk) of a unit cell. The dimensionless resistance is then

$$\psi = \frac{\kappa'^{3/2}(1-\kappa')^{1/2}}{1.8506}\Big\{1-0.3443\kappa'^{3/2}-0.1111\kappa'^{5/2}$$
$$-0.06588\kappa'^{7/2}-0.04808\kappa'^{9/2}-\cdots\Big\}^{-1} \tag{40}$$

and following Eq. (8), we have $\Phi=\psi/(1-\kappa')$.

D. Adiabatic Disks in a Hexagonal Array

The same corrective-iterative procedure used in the analysis of a square array of adiabatic disks can be applied to the case of a hexagonal array. In this case, the temperature components $T^{(0)}$, $\tilde{T}_H^{(0)}$, $T_H^{(3)}$, $T_H^{(5)}$, $T_H^{(3+3)}$, $T_H^{(7)}$, $T_H^{(3+5)}$, $T_H^{(5+3)}$, $T_H^{(9)}$, $T_H^{(9),6}$, and $T_H^{(3+3+3)}$ are needed. For an isolated disk, the temperature field $\tilde{T}_H^{(0)}$ can be easily obtained from $\tilde{T}_S^{(0)}$ by replacing the factor $[(2N+1)^2-1]$ with $[3N(N+1)]$. The components of higher order, except $T_H^{(9),6}$, can be obtained from the corresponding temperature distributions in the analysis of a square array of disks by replacing the factors $[(2N+1)^2]$ and $s_i(N)$ with $[3N(N+1)+1]$ and $h_i(N)$, respectively. For an isolated disk, $T_H^{(9),6}$ yields the disk temperature of

$$T_H^{(9),6}\big|_{\text{disk}} = -\frac{2}{\pi}(3N^2+3N+1)\frac{qR}{k}\cdot\frac{4}{3\pi}\left(\frac{\rho}{R}\right)^6$$
$$\times\left[1-\left(\frac{\rho}{R}\right)^2\right]^{1/2}\cos 6\phi\, h_{9,6}(N)\left(\frac{R}{\delta}\right)^9 \tag{41}$$

while the rest of the surface and the far-field region of the solid is at zero tempearture. The surface heat flux is given by

$$\begin{cases} q_H^{(9),6}\big|_{\text{disk}} = -\dfrac{2}{\pi}(3N^2+3N+1)q\cdot\dfrac{1001}{512} \\ \quad\times\left(\dfrac{\rho}{R}\right)^6\cos 6\phi\, h_{9,6}(N)\left(\dfrac{R}{\delta}\right)^9 \\ q_H^{(9),6}\big|_{\rho>R} = \dfrac{2}{\pi}(3N^2+3N+1)q \\ \quad\times\left[\mathcal{O}\left(\dfrac{R}{\rho}\right)^9\right]\cos 6\phi\, h_{9,6}(N)\left(\dfrac{R}{\delta}\right)^9 \end{cases} \tag{42}$$

Thus, after superposing the components $T^{(0)}$, $\tilde{T}_H^{(0)}$, $T_H^{(3)}$, $T_H^{(5)}$, $T_H^{(3+3)}$, $T_H^{(7)}$, $T_H^{(3+5)}$, $T_H^{(5+3)}$, $T_H^{(9)}$, $T_H^{(3+3+3)}$, and $T_H^{(9),6}$ at each of the disks inside the hexagon in Fig. 3, and letting $N\to\infty$, we obtain

$$\Delta T^* = \frac{2}{\pi}\frac{\bar{q}R}{k}\Xi_\rho^{1/2} + \frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{2H_3}{3\pi}\Xi_\rho^{1/2}\left(\frac{R}{\delta}\right)^3$$
$$+\frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{H_5}{\pi}\left(\frac{8}{5}\Xi_\rho^{1/2}-\frac{2}{3}\Xi_\rho^{3/2}\right)\left(\frac{R}{\delta}\right)^5$$
$$+\frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{4H_3^2}{9\pi^2}\Xi_\rho^{1/2}\left(\frac{R}{\delta}\right)^6+\frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{H_7}{\pi}$$
$$\times\left(\frac{30}{7}\Xi_\rho^{1/2}-\frac{10}{3}\Xi_\rho^{3/2}+\frac{2}{3}\Xi_\rho^{5/2}\right)\left(\frac{R}{\delta}\right)^7$$
$$+\frac{2}{\pi}\frac{\bar{q}R}{k}\cdot\frac{H_3H_5}{\pi^2}\left(\frac{28}{15}\Xi_\rho^{1/2}-\frac{4}{9}\Xi_\rho^{3/2}\right)\left(\frac{R}{\delta}\right)^8$$
$$+\frac{2}{\pi}\frac{\bar{q}R}{k}\left(\frac{H_9}{\pi}\left\{\frac{112}{9}\Xi_\rho^{1/2}-14\Xi_\rho^{3/2}+\frac{28}{5}\Xi_\rho^{5/2}-\frac{2}{3}\Xi_\rho^{7/2}\right\}\right.$$
$$\left.+\frac{8H_3^3}{27\pi^3}\Xi_\rho^{1/2}+4\frac{H_{9,6}}{3\pi}\Xi_\rho^{1/2}\left(\frac{\rho}{R}\right)^6\cos 6\phi\right)\left(\frac{R}{\delta}\right)^9+\mathcal{O}\left(\frac{R}{\delta}\right)^{10} \tag{43}$$

where

$$h_i(N) = 6\sum_{n=1}^{N}\frac{1}{n^i}+6\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\frac{1}{(m^2-mn+n^2)^{i/2}} \tag{44}$$

$$h_{9,6}(N) = 6\sum_{n=1}^{N}\frac{1}{n^9}+6\sum_{n=1}^{N-1}\sum_{m=n+1}^{N}\frac{1}{(m^2-mn+n^2)^{9/2}}$$
$$\times\left[1-\frac{27}{2}\frac{m^2n^2(m-n)^2}{(m^2-mn+n^2)^3}\right] \tag{45}$$

$$H_i = \lim_{N\to\infty}h_i(N),\qquad H_{9,6}=\lim_{N\to\infty}h_{9,6}(N) \tag{46}$$

Here, ΔT^* is the difference between the temperature of the isothermal region of the solid surface and the temperature of the disks arranged in an infinite hexagonal array. This corresponds to the condition of disks adiabatic up to $\mathcal{O}(R/\delta)^9$. In Eq. (43) $\bar{q}$ is the average heat flux into the solid at its surface. Averaging ΔT^* over a disk, we obtain

$$\overline{\Delta T^*} = \frac{2}{\pi}\frac{\bar{q}R}{k}\left\{\frac{2}{3}+\frac{4H_3}{9\pi}\left(\frac{R}{\delta}\right)^3+\frac{4H_5}{5\pi}\left(\frac{R}{\delta}\right)^5\right.$$
$$+\frac{8H_3^2}{27\pi^2}\left(\frac{R}{\delta}\right)^6+\frac{12H_7}{7\pi}\left(\frac{R}{\delta}\right)^7+\frac{16H_3H_5}{15\pi^2}\left(\frac{R}{\delta}\right)^8$$
$$\left.+\left(\frac{16H_3^3}{81\pi^3}+\frac{112H_9}{27\pi}\right)\left(\frac{R}{\delta}\right)^9+\mathcal{O}\left(\frac{R}{\delta}\right)^{10}\right. \tag{47}$$

The difference between the temperature of the isothermal region and the average temperature of the entire surface of the solid is given by

$$\Delta T = \frac{\pi R^2}{\delta^2\sqrt{3}/2}\overline{\Delta T^*} = \kappa'\overline{\Delta T^*} \tag{48}$$

Upon substituting Eq. (48) into Eq. (8) and inserting the numerical values of $H_3=11.0342$, $H_5=6.76190$, $H_7=6.19524$, and $H_9=6.05695$ into the resulting equation, we obtain

$$\psi = \frac{\kappa'^{3/2}(1-\kappa')^{1/2}}{1.8506}\Big\{1-0.3389\kappa'^{3/2}-0.1031\kappa'^{5/2}$$
$$-0.05577\kappa'^{7/2}-0.03637\kappa'^{9/2}-\cdots\Big\}^{-1} \tag{49}$$

and again, $\Phi=\psi/(1-\kappa')$.

III. Results and Discussion

The principal objective of this paper is to observe the variation of thermal constriction resistance with the contact geometry. Since the adiabatic circular gaps represent quite a different geometry from the isothermal circular contacts, it would be worthwhile to compare these two. We first need to estimate the error in the series in Eqs. (40) and (49). For $\kappa' = 1 - \kappa = 0.7$, which corresponds to a square array of nearly touching disks, using the first four terms and the first five terms of the series of Eq. (40) results in the difference of less than 2%; for Eq. (49), the difference is only 1% with $\kappa' = 0.7$ and about 2% with $\kappa' = 0.8$. Thus, for practical purposes, the level of accuracy in the Eqs. (40) and (49) is sufficient. The actual error would be better represented by the $\mathcal{O}(\kappa'^{11/2})$ terms which will be even lower than these figures.

Figure 4 shows the resistance ψ of a square array and a hexagonal array of adiabatic disks [Eqs. (40) and (49)]. For comparison purposes, we also include the resistance of isothermal disks on the otherwise insulated surface of a semi-infinite solid[33] and the resistance of a circular cell (an isothermal disk on a laterally insulated semi-infinite cylinder).[29] Also included in Fig. 4 is the resistance predicted by the approximate formula of Eq. (52); we will discuss this in detail later. In Fig. 5, we plot the corresponding resistance Φ of isothermal and adiabatic disks arranged in a square or hexagonal array.

From Figs. 4 and 5, a few observations can be made. For both isothermal and adiabatic cases, a hexagonal array yields a lower resistance than a square array. This is because the unit cells of a hexagonal array resemble a circle more than those of a square array. For adiabatic disks, a square array and a hexagonal array yield a practically equal resistance. In fact, at an area fraction of contact of $\kappa = 0.3$ (or $\kappa' = 0.7$), the relative difference is only 1.6%; furthermore, this difference decreases to zero with decreasing κ'. For isothermal disks, the relative difference between a square array and a hexagonal array increases from 0 to about 20% as the area fraction of contact (or disk area fraction) increases from 0 to 0.7. In other words, the relative difference in resistance between a square array and a hexagonal array increases with disk area fraction κ'. Furthermore, the difference for adiabatic disks is smaller than that for isothermal disks. These two points can be understood intuitively. First, as κ' increases, the disk of a unit cell of the array approaches the edges of the cell; hence, the effect of the shape of the unit cell becomes more pronounced. For the second point, the interpretation is as follows: in the case of adiabatic disks, the (originally parallel) lines of heat flow from the warmer solid toward a disk bend outward as they approach the disk and enter the cooler solid at that part of the interface immediately around the disk. On the other hand, all the lines of heat flow outside the contact area in an isothermal-disk unit cell bend inward and enter the cooler solid at the disk. Consequently, the shape of the unit cells has a greater effect on the constriction resistance in the case of isothermal disks.

It would be useful to define a length scale that can be applied to a large range of contact geometries, analogous to the hydraulic diameter for the Graetz problem. In an attempt to obtain such a length scale, we have tried to find a parameter that will yield a nearly uniform dimensionless resistance for the cases worked out in the current analysis as well as the isothermal disk cases.[33] We first look for a single length scale that leads to unit dimensionless resistance for $\kappa = 0$ and $\kappa = 1$. In the limit of $\kappa \to 0$, for isothermal circular contacts, the resistance is exactly proportional to the reciprocal of the perimeter P of the contact area. Therefore, the perimeter can be considered to be a suitable length scale. On the other hand, in the limit of $\kappa \to 1$, Eqs. (38), (47), and (48) show the resistance being proportional to $\kappa'^2/R = (1-\kappa)^2/R$. In view of this observation, we can infer that $P/(1-\kappa)^2$ is a suitable length scale. In fact, it is consistent with the scaling derived from the limit of $\kappa \to 0$. If we now define a length scale $L = P/[2\pi(1-\kappa)^2]$, then the nondimensional resistance

$$\Psi = 4kL\psi^* \tag{50}$$

is equal to unity at $\kappa = 0$. In the other limit of $\kappa = 1$, we have $\Psi = 16/(3\pi^2) = 0.5404$. In view of this, we modify L further by a factor that is equal to unity at $\kappa = 0$ and equal to 0.5404 at $\kappa = 1$. Choosing a linear fit in $\kappa^{1/2}$ between these two numbers, we obtain the factor $\{1 - [1 - 16/(3\pi^2)]\kappa^{1/2}\}$. Thus we define a length scale

$$L = \frac{P}{2\pi(1-\kappa)^2\left\{1 - \left[1 - 16/(3\pi^2)\right]\kappa^{1/2}\right\}} \tag{51}$$

Using this length scale in Eq. (50), we find that the dimensionless resistance Ψ lies between 0.8 and 1.1 for square or hexagonal arrays of adiabatic or isothermal circular disks. It is easy to see from here that in general the resistance is approximately given by

$$\psi^* \simeq \frac{\pi(1-\kappa)^2\left\{1 - \left[1 - 16/(3\pi^2)\right]\kappa^{1/2}\right\}}{2kP} \tag{52}$$

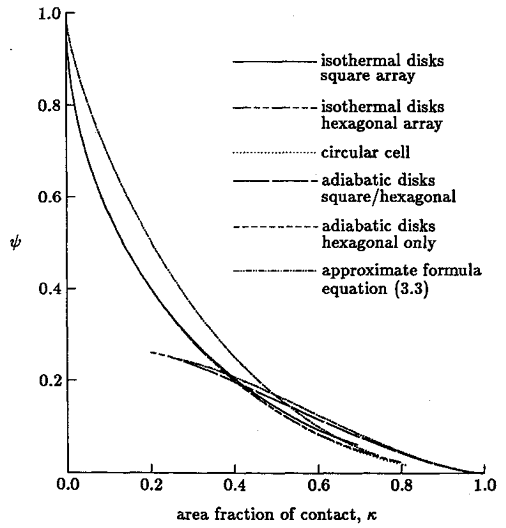

Fig. 4 Dimensionless resistance ψ of adiabatic and isothermal disks arranged in a square or hexagonal array; also shown are the resistance of a circular cell and the approximate formula of Eq. (52).

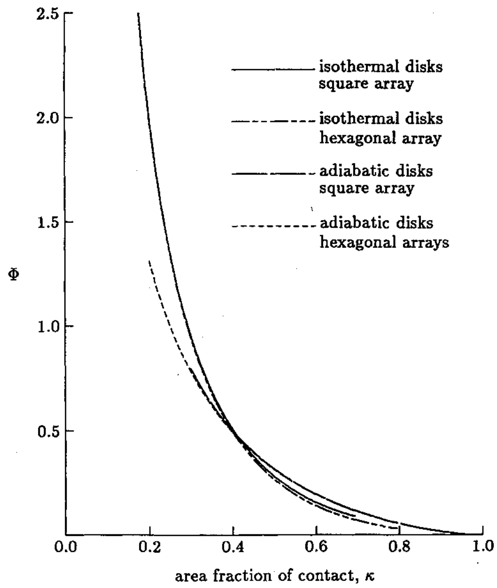

Fig. 5 Dimensionless resistance Φ of adiabatic and isothermal disks arranged in a square or hexagonal array.

or

$$\Phi^* \simeq \frac{\pi(1-\kappa)^2\left\{1-\left[1-16/(3\pi^2)\right]\kappa^{1/2}\right\}}{2kp} \tag{53}$$

where p is the perimeter per unit area of interface. In order to obtain the resistance, one simply has to calculate P or p for the particular geometry in consideration and substitute the number into Eqs. (52) or (53). In Fig. 4, we compare the approximate resistance [Eq. (52)] to that predicted by the exact formulas for both cases of isothermal and adiabatic disks. For isothermal disks, the exact and approximate formulas agree to within about 20%; for adiabatic disks, the corresponding figure is 7%. We do need to point out that this generalization is based exclusively on the developments dealing with the spatially periodic arrays of identical circular disks. Projections of this result to other contact shapes cannot be immediately made without further tests.

Finally, we note that the solution of this study is also applicable to cases of circular regions with uniform flux on an otherwise isothermal surface. The corresponding single-disk solution was obtained by Lemczyk and Yovanovich.[17]

Appendix: Corrective Terms for Heat Flux

$$\begin{cases} q_S^{(3)}\,|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{1}{3}s_3(N)\left(\dfrac{R}{\delta}\right)^3 \\ q_S^{(3)}\,|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{2}{3\pi}\left[\dfrac{1}{3}\left(\dfrac{R}{\rho}\right)^3+\dfrac{3}{10}\left(\dfrac{R}{\rho}\right)^5 \right. \\ \qquad \left. +\dfrac{15}{56}\left(\dfrac{R}{\rho}\right)^7+\cdots\right]s_3(N)\left(\dfrac{R}{\delta}\right)^3 \end{cases}$$

$$T_S^{(3)}\,|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{2}{3\pi}\Xi_\rho^{1/2}s_3(N)\left(\frac{R}{\delta}\right)^3$$

$$\begin{cases} q_S^{(5)}\,|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\left[\dfrac{3}{10}+\dfrac{3}{4}\left(\dfrac{\rho}{R}\right)^2\right]s_3(N)\left(\dfrac{R}{\delta}\right)^5 \\ q_S^{(5)}\,|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{1}{\pi} \\ \qquad \times\left[\dfrac{2}{5}\left(\dfrac{R}{\rho}\right)^3+\dfrac{69}{175}\left(\dfrac{R}{\rho}\right)^5+\cdots\right]s_5(N)\left(\dfrac{R}{\delta}\right)^5 \end{cases}$$

$$T_S^{(5)}\,|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{1}{\pi}\left(\frac{8}{5}\Xi_\rho^{1/2}-\frac{2}{3}\Xi_\rho^{3/2}\right)s_5(N)\left(\frac{R}{\delta}\right)^5$$

$$\begin{cases} q_S^{(3+3)}\,|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{2}{9\pi}s_3^2(N)\left(\dfrac{R}{\delta}\right)^6 \\ q_S^{(3+3)}\,|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{4}{9\pi^2} \\ \qquad \times\left[\dfrac{1}{3}\left(\dfrac{R}{\rho}\right)^3+\dfrac{3}{10}\left(\dfrac{R}{\rho}\right)^5+\cdots\right]s_3^2(N)\left(\dfrac{R}{\delta}\right)^6 \end{cases}$$

$$T_S^{(3+3)}\,|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{4}{9\pi^2}\Xi_\rho^{1/2}s_3^2(N)\left(\frac{R}{\delta}\right)^6$$

$$\begin{cases} q_S^{(7)}\,|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\left[\dfrac{15}{56}+\dfrac{15}{8}\left(\dfrac{\rho}{R}\right)^2 \right. \\ \qquad \left. +\dfrac{75}{64}\left(\dfrac{\rho}{R}\right)^4\right]s_7(N)\left(\dfrac{R}{\delta}\right)^7 \\ q_S^{(7)}\,|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{1}{\pi}\left[\dfrac{6}{7}\left(\dfrac{R}{\rho}\right)^3+\cdots\right]s_7(N)\left(\dfrac{R}{\delta}\right)^7 \end{cases}$$

$$T_S^{(7)}\,|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{1}{\pi}$$
$$\times\left(\frac{30}{7}\Xi_\rho^{1/2}-\frac{10}{3}\Xi_\rho^{3/2}+\frac{2}{3}\Xi_\rho^{5/2}\right)s_7(N)\left(\frac{R}{\delta}\right)^7$$

$$\begin{cases} q_S^{(7),4}\,|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\left[\dfrac{105}{64}\left(\dfrac{\rho}{R}\right)^4\cos 4\phi\right] \\ \qquad \times s_{7,4}(N)\left(\dfrac{R}{\delta}\right)^7 \\ q_S^{(7),4}\,|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\left[\mathcal{O}\left(\dfrac{R}{\rho}\right)^7\right]\cos 4\phi\, s_{7,4}(N)\left(\dfrac{R}{\delta}\right)^7 \end{cases}$$

$$T_S^{(7),4}\,|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{4}{3\pi}\left(\frac{\rho}{R}\right)^4$$
$$\times\,\Xi_\rho^{1/2}\cos 4\phi\, s_{7,4}(N)\left(\frac{R}{\delta}\right)^7$$

$$\begin{cases} q_S^{(3+5)}\,|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{2}{3\pi}\left[\dfrac{3}{10}+\dfrac{3}{4}\left(\dfrac{\rho}{R}\right)^2\right] \\ \qquad \times s_3(N)s_5(N)\left(\dfrac{R}{\delta}\right)^8 \\ q_S^{(3+5)}\,|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{2}{3\pi^2} \\ \qquad \times\left[\dfrac{2}{5}\left(\dfrac{R}{\rho}\right)^3+\dfrac{69}{175}\left(\dfrac{R}{\rho}\right)^5+\cdots\right]s_3(N)s_5(N)\left(\dfrac{R}{\delta}\right)^8 \end{cases}$$

$$T_S^{(3+5)}\,|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{2}{3\pi^2}\left(\frac{8}{5}\Xi_\rho^{1/2}-\frac{2}{3}\Xi_\rho^{3/2}\right)$$
$$\times\, s_3(N)s_5(N)\left(\frac{R}{\delta}\right)^8$$

$$\begin{cases} q_S^{(5+3)}\,|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{2}{5\pi^2}s_3(N)s_5(N)\left(\dfrac{R}{\delta}\right)^8 \\ q_S^{(5+3)}\,|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{4}{5\pi^2} \\ \qquad \times\left[\dfrac{1}{3}\left(\dfrac{R}{\rho}\right)^3+\dfrac{3}{10}\left(\dfrac{R}{\rho}\right)^5+\cdots\right]s_3(N)s_5(N)\left(\dfrac{R}{\delta}\right)^8 \end{cases}$$

$$T_S^{(5+3)}\,|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{4}{5\pi^2}\Xi_\rho^{1/2}s_3(N)s_5(N)\left(\frac{R}{\delta}\right)^8$$

$$\begin{cases} q_S^{(9)}|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\left[\dfrac{35}{144}+\dfrac{105}{32}\left(\dfrac{\rho}{R}\right)^2 \right. \\ \left. \quad +\dfrac{735}{128}\left(\dfrac{\rho}{R}\right)^4+\dfrac{1225}{768}\left(\dfrac{\rho}{R}\right)^6\right] s_9(N)\left(\dfrac{R}{\delta}\right)^9 \\ q_S^{(9)}|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{1}{\pi}\left[\dfrac{56}{27}\left(\dfrac{R}{\rho}\right)^3+\cdots\right] s_9(N)\left(\dfrac{R}{\delta}\right)^9 \end{cases}$$

$$T_S^{(9)}|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{1}{\pi} \times\left(\frac{112}{9}\Xi_\rho^{1/2}-14\Xi_\rho^{3/2}+\frac{28}{5}\Xi_\rho^{5/2}-\frac{2}{3}\Xi_\rho^{7/2}\right)s_9(N)\left(\frac{R}{\delta}\right)^9$$

$$\begin{cases} q_S^{(9),4}|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q \\ \quad \times\left[\dfrac{693}{128}\left(\dfrac{\rho}{R}\right)^4+\dfrac{693}{256}\left(\dfrac{\rho}{R}\right)^6\right]\cos 4\phi\, s_{9,4}(N)\left(\dfrac{R}{\delta}\right)^9 \\ q_S^{(9),4}|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\left[\mathcal{O}\left(\dfrac{R}{\rho}\right)^7\right]\cos 4\phi\, s_{9,4}(N)\left(\dfrac{R}{\delta}\right)^9 \end{cases}$$

$$T_S^{(9),4}|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{1}{\pi} \times\left(\frac{32}{5}\Xi_\rho^{1/2}-\frac{4}{3}\Xi_\rho^{3/2}\right)\left(\frac{\rho}{R}\right)^4\cos 4\phi\, s_{9,4}(N)\left(\frac{R}{\delta}\right)^9$$

$$\begin{cases} q_S^{(3+3+3)}|_{\text{disk}} = -\dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{4}{27\pi^2}s_3^3(N)\left(\dfrac{R}{\delta}\right)^9 \\ q_S^{(3+3+3)}|_{\rho>R} = \dfrac{2}{\pi}(2N+1)^2 q\cdot\dfrac{8}{27\pi^3} \\ \quad \times\left[\dfrac{1}{3}\left(\dfrac{R}{\rho}\right)^3+\dfrac{3}{10}\left(\dfrac{R}{\rho}\right)^5+\ldots\right]s_3^3(N)\left(\dfrac{R}{\delta}\right)^9 \end{cases}$$

$$T_S^{(3+3+3)}|_{\text{disk}} = -\frac{2}{\pi}(2N+1)^2\frac{qR}{k}\cdot\frac{8}{27\pi^3}\Xi_\rho^{1/2}s_3^3(N)\left(\frac{R}{\delta}\right)^9$$

Acknowledgments

The authors are very grateful to the U.S. National Science Foundation for the support of this work under Grant Nos. CBT-8351432 and INT-8807482. Thanks are also due to the Department of Applied Mathematics and Theoretical Physics at Cambridge University for the availability of their research facilities to the second author (S. Sadhal), who was on a sabbatical leave there (July 5, 1988–July 18, 1989).

References

[1]Madhusudana, C. V., and Fletcher, L. S., "Contact Heat Transfer—The Last Decade," *AIAA Journal,* Vol. 24, No. 3, 1986, pp. 510–523.

[2]Yovanovich, M. M., "Recent Developments in Thermal Contact, Gap and Joint Conductance Theories and Experiment," *Proceedings of the Eighth International Heat Transfer Conference,* Vol. 1, edited by C. L. Tien et al., Hemisphere, Washington, DC, 1986, pp. 35–45.

[3]Fletcher, L. S., "Recent Developments in Contact Conductance Heat Transfer," *Journal of Heat Transfer,* Vol. 110, No. 4B, 1988, pp. 1059–1070.

[4]Weber, H., "Über die Besselschen Functionen und ihre Anwendung auf die Theorie der elektrishcen Ströme," *Journal für die reine und angewandte Mathematik,* Vol. 75, 1873, pp. 75–105.

[5]Jackson, J. D., *Classical Electrodynamics,* Wiley, New York, 1962, pp. 89–93.

[6]Heasley, J. H., "Transient Heat Flow between Contacting Solids," *International Journal of Heat and Mass Transfer,* Vol. 8, No. 1, 1965, pp. 147–154.

[7]Schneider, G. E., Strong, A. B., and Yovanovich, M. M., "Transient Thermal Response of Two Bodies Communicating through a Small Circular Contact Area," *International Journal of Heat and Mass Transfer,* Vol. 20, No. 4, 1977, pp. 301–308.

[8]Sadhal, S. S., "Transient Thermal Response of Two Solids in Contact over a Circular Disk," *International Journal of Heat and Mass Transfer,* Vol. 23, No. 5, 1980, pp. 731–733.

[9]Beck, J. V., and Keltner, N. R., "Transient Thermal Contact of Two Semi-Infinite Bodies over a Circular Area," *Spacecraft Radiative Transfer and Temperature Control,* edited by T. E. Horton, Vol. 83, Progress in Astronautics and Aeronautics, AIAA, New York, 1982, pp. 61–82.

[10]Thomas, P. H., "Some Conduction Problems in the Heating of Small Areas on Large Solids," *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 10, Pt. 4, 1957, pp. 482–493.

[11]Beck, J. V., "Large Time Solutions for Temperatures in a Semi-Infinite Body with a Disk Heat Source," *International Journal of Heat and Mass Transfer,* Vol. 24, No. 1, 1981, pp. 155–164.

[12]Norminton, E. J., and Blackwell, J. H., "Transient Heat Flow from Constant Temperature Spheroids and the Thin Circular Disk," *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 17, Pt. 1, 1964, pp. 65–72.

[13]Blackwell, J. H., "Transient Heat Flow from a Thin Circular Disk—Small-Time Solution," *Journal of the Australian Mathematical Society,* Vol. 14, 1972, pp. 433–442.

[14]Strong, A. B., Schneider, G. E., and Yovanovich, M. M., "Thermal Constriction Resistance of a Disk with Arbitrary Heat Flux," *Heat Transfer with Thermal Control Applications,* edited by M. M. Yovanovich, Vol. 39, Progress in Astronautics and Aeronautics, AIAA, New York, 1975, pp. 65–78.

[15]Marder, B. M., and Keltner, N. R., "Heat Flow from a Disk by Separation of Variables," *Numerical Heat Transfer,* Vol. 4, No. 4, 1981, pp. 485–497.

[16]Gladwell, G. M. L., Barber, J. R., and Olesiak, Z., "Thermal Problems with Radiation Boundary Conditions," *Quarterly Journal of Mechanics and Applied Mathematics,* Vol. 36, Pt. 3, 1983, pp. 387–401.

[17]Lemczyk, T. F., and Yovanovich, M. M., "Thermal Constriction Resistance with Convective Boundary Conditions—1. Half-Space Contacts," *International Journal of Heat and Mass Transfer,* Vol. 31, No. 9, 1988, pp. 1861–1872.

[18]Yovanovich, M. M., "Thermal Constriction Resistance of Contacts on a Half-Space: Integral Formulation," *Radiative Transfer and Thermal Control,* edited by A. M. Smith, Vol. 49, Progress in Astronautics and Aeronautics, AIAA, New York, 1976, pp. 397–418.

[19]Schneider, G. E., "Thermal Resistance due to Arbitrary Dirichlet Contacts on Half-Space," *Thermophysics and Thermal Control,* edited by R. Viskanta, Vol. 65, Progress in Astronautics and Aeronautics, AIAA, New York, 1979, pp. 103–119.

[20]Dundurs, J., and Panek, C., "Heat Conduction between Bodies with Wavy Surfaces," *International Journal of Heat and Mass Transfer,* Vol. 19, No. 7, 1976, pp. 731–736.

[21]Sadhal, S. S., "Unsteady Heat Flow between Solids with Partially Contacting Interface," *Journal of Heat Transfer,* Vol. 103, No. 1, 1981, pp. 32–35.

[22]Greenwood, J. A., "Constriction Resistance and the Real Area of Contact," *British Journal of Applied Physics,* Vol. 17, Dec. 1966, pp. 1621–1632.

[23]Beck, J. V., "Effects of Multiple Sources in the Contact Conductance Theory," *Journal of Heat Transfer,* Vol. 101, No. 1, 1979, pp. 132–136.

[24]Sadhal, S. S., "Exact Solutions for the Steady and Unsteady Diffusion Problems for a Rectangular Prism: Cases of Complex Neumann Conditions," American Society of Mechanical Engineers, New York, Paper 84-HT-83, 1984.

[25]Negus, K. J., and Yovanovich, M. M., "Application of the Method of Optimized Images to Steady Three-Dimensional Conduction Problems," American Society of Mechanical Engineers, New York, Paper 84-WA/HT-110, 1984.

[26]Negus, K. J., Yovanovich, M. M., and Devaal, J. W., "Development of Thermal Constriction Resistance for Anisotropic Rough Surfaces by the Method of Infinite Images," American Society of Me-

chanical Engineers, New York, Paper 85-HT-17, 1985.
[27]Hunter, A., and Williams, A., "Heat Flow across Metallic Joints—The Constriction Alleviation Factor," *International Journal of Heat and Mass Transfer,* Vol. 12, No. 4, 1969, pp. 524-526.
[28]Leppington, F. G., and Levine, H., "Some Axially Symmetric Potential Problems," *Proceedings of the Edinburgh Mathematical Society,* Edinburgh Mathematical Society, Edinburgh, Scotland, UK, Vol. 18, 1972, pp. 55-76.
[29]Negus, K. J., and Yovanovich, M. M., "Constriction Resistance of Circular Flux Tubes with Mixed Boundary Conditions by Linear Superposition of Neumann Solutions," American Society of Mechanical Engineers, New York, Paper 84-HT-84, 1984.
[30]Gladwell, G. M. L., and Lemczyk, T. F., "Thermal Constriction Resistance of a Contact on a Circular Cylinder with Mixed Convective Boundaries," *Proceedings of the Royal Society of London,* Royal Society of London, London, Vol. A420, 1988, pp. 323-354.
[31]Batchelor, G. K., "Sedimentation in a Dilute Dispersion of Spheres," *Journal of Fluid Mechanics,* Vol. 52, Pt. 2, 1972, pp. 245-268.
[32]Negus, K. J., Yovanovich, M. M., and Beck, J. V., "On the Nondimensionalization of Constriction Resistance for Semi-Infinite Heat Flux Tubes," *Journal of Heat Transfer,* Vol. 111, No. 3, 1989, pp. 804-807.
[33]Tio, K.-K., "Transport Problems with Spatially Periodic Mixed Interface Conditions," Ph.D. Thesis, University of Southern California, Los Angeles, 1990.